Scattering of sine-Gordon breathers on a potential well

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We analyze the scattering of the classical sine-Gordon breathers on a square potential well. We show that the scattering process depends not only on the vibration frequency of the breather and its incoming speed but also on its phase as well as the depth and width of the well. We show that the breather can pass through the well and exit with a speed different, sometimes larger, from the initial one. It can also be trapped and very slowly decay inside the well or bounce out of the well and go back to where it came from. We also show that the breather can split into a kink–antikink pair when it hits the well.

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I. INTRODUCTION

The sine-Gordon model is probably the most studied integrable model. One of the reasons for this is that it describes a large variety of physical systems ranging from the Josephson effects [1], particle physics [2], information transport in microtubules [3], nonlinear optics [4], crystal dislocations [5], and ferromagnets [6].

In the inhomogeneous version of the sine-Gordon model, the coefficient in front of the potential becomes a function of x. In this paper we consider a square-well potential where the potential coefficient is one everywhere except in a small region of finite width where it takes a smaller value. In physical applications the potential coefficient is usually determined by one of the physical properties of the system. For the propagation of crystal dislocations, the potential coefficient is a function of the stiffness of the lattice [5,7], while for the Bloch-wall motion in a magnetic crystal it is related to the magnetization amplitude of the crystal and its anisotropy energy parameter [7]. For ultrashort optical pulses in a two-level atomic system the potential coefficient is directly proportional to the average state density of the two-level system [7]. For the soliton dynamics in easy-plane ferromagnets the potential coefficient is directly proportional to the external magnetic field applied to the ferromagnets [6].

The square well we have chosen thus corresponds to a system where those physical properties take two different values and where the transition region between these values is very small compared to the size of the inhomogeneity and the size of the sine-Gordon kink or breather.

From a mathematical point of view the sine-Gordon model is interesting for several reasons. First of all it is Lorentz invariant. This means that any stationary solution can be boosted to any speed $v<1$, a key property to perform any scattering. Moreover as any finite-energy solution, say, describing a kink, of the sine-Gordon model corresponds to a mapping from the circle into itself; each solution is characterized by a topological charge taking integer values. By conventions, solitonic solutions with a negative topological charge are called antikinks. In principle, a kink and an antikink could annihilate with each other, but because of the integrability of the model, instead, they scatter or form bound states which are called breathers. Breathers have been extensively studied and they are known, for example, to scatter with each other elastically, like kinks or antikinks.

Our system with a potential well can be viewed as three sine-Gordon systems (with different coupling constants) tied together with simple boundary conditions involving continuity of the fields and its derivatives at the "junctions."

When phrased like this one can compare it to the systems involved in various studies of sine-Gordon fields with defects. Among such studies one finds many papers [8] which have considered effects of inserting defects into sine-Gordon fields. However, most of these studies concentrated their attention on preserving integrability of the resultant theories. This, in fact, has always required the imposition of boundary conditions involving time derivatives of the fields. Although this is very interesting from the mathematical point of view and has led to many interesting results, it is not very physical. Our approach is more physical and, as we know from the work in [8], does violate integrability. But being more physical it is interesting to see what its effects are.

In recent years, the scattering of a classical kink on a square-well potential was studied by several research groups [9–12]. The main results demonstrated that a kink sent toward the well with a speed larger than some critical value, which depends on the potential well, always goes through the well and comes out, at the other side, with a speed which is smaller than its original value. For smaller speeds, the kink ends up being trapped inside the well, except for a few very special values of the initial speed, just below the critical value, for which the kink bounces back and returns from where it came from.

In this paper we investigate how the breather scatters on the same well. The scattering of a breather in inhomogeneous systems is not new. A few years ago Zhang [13] studied the scattering of the breather on a $1/\cosh^2$ potential. For a very narrow well, the square well is very similar to the potential studied by him, while for a wide well, it is quite different, as in our case the nonintegrability is generated by only two points—the edges of the well. This means that breathers and kinks can freely propagate inside a well that is larger than their own size. This, as we will show, has a significant influence on the scattering properties of the breather.

The homogeneous sine-Gordon breathers are very special solutions of the sine-Gordon model which correspond to

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bound states of a kink and an antikink. They are time dependent and describe local oscillations of the field. They depend on one parameter which simultaneously sets the amplitude and the frequency of oscillation of the breather. In the limit of an infinite period, a breather corresponds to the scattering of a kink and an antikink, while for the smallest amplitude, it describes a small localized vacuum excitation. Unlike kink or antikink solutions, breathers asymptotically go to the same vacuum values at both spatial infinities.

In what follows, we present a detailed study of the scattering of the breather on a square well, analyzing the dependence on all the parameters which influence this scattering. As Zhang, we have observed several modes of scattering: transmission or reflection of the breather, trapping of the breather, and splitting of the breather into a kink and an antikink. We find a strong dependence of the scattering modes on the breather oscillation phase and then discuss the relative occurrence of these different modes when one varies the parameters of the model.

II. SINE-GORDON MODEL WITH A POTENTIAL WELL

The classical sine-Gordon model with a square potential well is defined by the following Lagrangian:

$$\mathcal{L} = \int \frac{1}{2}\{f_t^2 - f_x^2 - 2k(1-\alpha)[1-\cos(f)]dx\}, \tag{1}$$

where

$$\alpha = a \qquad -L/2 < x < L/2, \tag{2}$$

$$\alpha = 0 \quad \text{elsewhere}. \tag{3}$$

When a is positive, the potential is thus a square well of width L and depth $|a|$.

In the dimension-full version of the equation, the parameter k corresponds to the strength of the potential term. In the adimensional form of the equation, it can be set to 1 by rescaling x and t. In all our simulations, we have taken $k=1$, but it is convenient to keep this parameter to describe the breather analytically so that one can easily obtain approximate expressions for kinks and breathers, as well as their energies, inside a large well by taking their expression for the homogeneous equation and replacing k by $k(1-a)$. Note also that t has been rescaled so that the speed of waves is 1.

The equation of motion is given by

$$f_{tt} - f_{xx} + k(1-\alpha)\sin f = 0. \tag{4}$$

The scattering of a kink on a square-well potential was studied in [9–11] where it was shown that at small speeds, the kink becomes trapped by the well while at large speeds it goes through the well but looses some energy through radiation and thus exits from the well with a speed smaller than the initial one. The speed above which the kink can escape from the well was called the critical velocity.

For velocities in a few very narrow ranges of incoming speeds, just below the critical velocity, it was observed that the kink does neither go through the well nor get trapped in it, but instead bounces out of the well and returns to where it came from. Thus we have a reflection.

After studying the scattering of a kink on the square well, it is natural to ask what happens to a breather sent toward a similar well. A breather which moves at speed v is described by [14]

$$f(x,t) = 4\ \mathrm{atan}\left[\frac{\sin(\omega\sqrt{k}t')\sqrt{1-\omega^2}}{\omega\cosh(\sqrt{1-\omega^2}\sqrt{k}x')}\right], \tag{5}$$

where $x'=\frac{x-vt}{\sqrt{1-v^2}}$ and $t'=\frac{t-vx}{\sqrt{1-v^2}}$. This $f(x,t)$ is a solution of the pure sine-Gordon equation, i.e., Eq. (4) when $\alpha=0$.

The energy of the breather can be easily calculated and is given by $E=16\sqrt{k}\frac{\sqrt{(1-\omega^2)}}{\sqrt{(1-v^2)}}$. Notice that in our units, the energy of a kink is equal to $8\sqrt{k}$. It is well known that the breather is a bound state of a kink and an antikink and thus the energy of the breather is less than the energy of a kink and an antikink infinitely separated (i.e., $16\sqrt{k}$).

A stationary breather is a periodic function of time of period $T=2\pi/(\omega\sqrt{k})$. For small values of ω, the period of the breather is thus very large and, at the apex of the oscillations, the kink and the antikink are well separated. When ω is nearly 1, the period is slightly larger than $2\pi/\sqrt{k}$, the maximum amplitude for the breather is small, and the kink and antikink never really separate from each other.

The scattering of a breather on the well depends on several parameters: the breather parameter ω, the incoming speed v, the width L, and the depth a of the well, as well as the phase of the breather when it hits the well. As the breather is an extended object, the scattering time cannot really be defined precisely and so it is difficult to accurately determine the phase of the breather at the time of the scattering. Nevertheless it is straightforward to show that, in the well frame, the distance traveled by the breather outside the well, i.e., when $a=0$, during one period of oscillation is equal to $d=2\pi v/[\omega\sqrt{k(1-v^2)}]$. To cover the full set of breather phases all we have to do is to put the breather initially at several positions within the range $[-x_1-d,-x_1]$, where x_1 must be sufficiently far away from the edge of the well so that, initially, the breather does not overlap with it.

To solve Eq. (4) numerically, we have used the fourth-order Runge-Kutta method for the time evolution and a second-order finite-difference scheme to discretize the spatial derivatives. We used grids extending from, at least, −60 to 60 with $dx=0.01$ and $dt=dx/4$. The grid size was chosen so that the distance between the breathers or the kinks and the edge of the grid was always at least 20 during the simulations.

The boundary conditions for the fields were set as follows: the field f and its time derivative f_t were set to 0 on the last lattice points at both ends of the grid. We also had some absorption at the edges of the grid which was introduced by adding the term $\gamma(x)f_t$ on the left-hand side of Eq. (4) with $\gamma(x)=0.1$ when $-60\leq x<55$ or $55<x\leq 60$ and $\gamma(x)=0$ elsewhere. This absorbed the waves generated by the breathers when they scatter on the well.

To investigate the dependence of the scattering on the parameters L, a, v, and ω we have systematically scanned the full range of the breather phase by varying the initial position of the breather in the range $[-x_1-d,-x_1]$ by step $dx=0.02$ (we have thus used over 300 values of the phase for each parameter set). We have then counted the number of

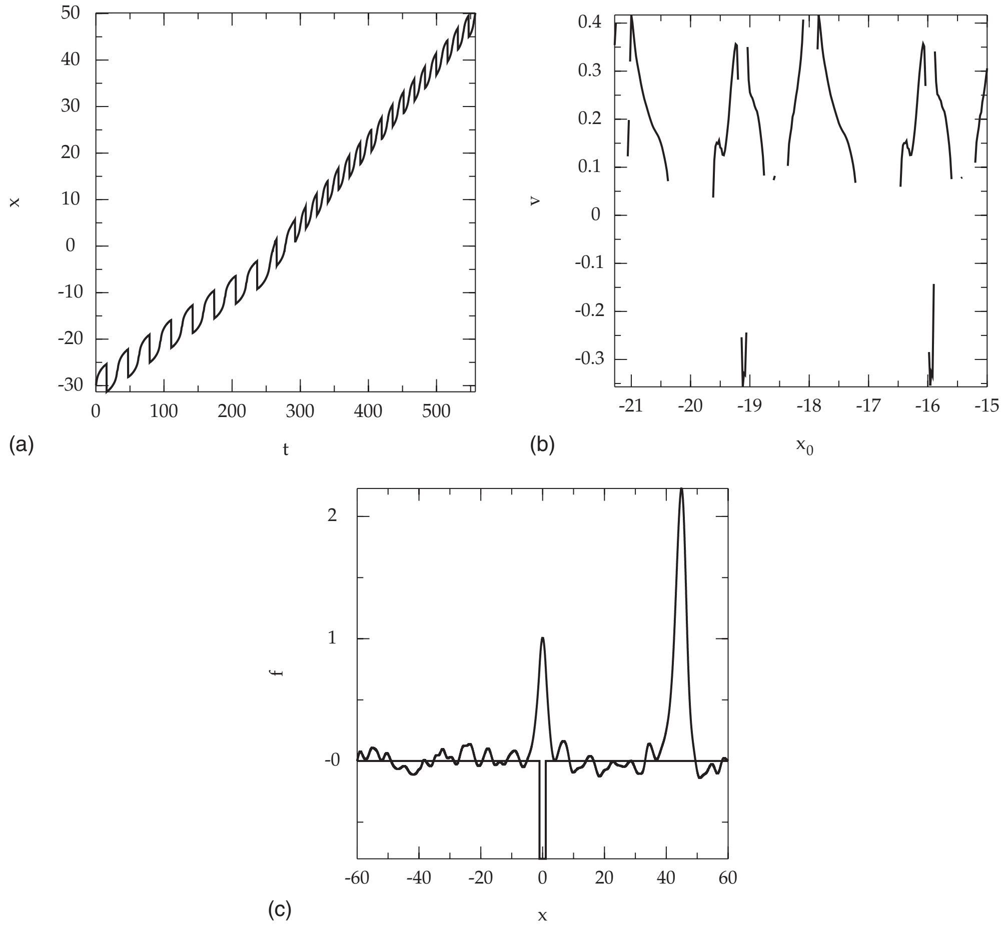

FIG. 1. (a) Breather position for a well with $L=2$, $a=0.2$, $v=\omega=0.1$, and $x_0=-29.92$. (b) Breather outgoing speed for a well with $L=10$, $a=0.2$, $v=\omega=0.1$ as a function of the initial position (phase) x_0. The gaps correspond to other types of scattering. (c) A solution profile for $L=2$, $a=0.8$, $v=\omega=0.3$, and $x_0=-15.5$. The breather is split into two: one is ejected from the well while the other one is trapped inside it.

times each type of scattering has taken place and compared their relative occurrences.

Before we analyze the dependence of the scattering on these parameters we first describe the different phenomena that we have observed, that is, transmission, trapping, backward, and forward scattering, as well as backward splitting. Initially, we consider the case of $v=\omega$ so that the energy of the breather equals the energy of a kink and an antikink. This is the critical case where the kink and antikink cannot split outside the well. So, unless otherwise stated, in the discussions below it is assumed that $v=\omega$. Later we report on what happens when $v\neq\omega$.

A. Breather transmission

When a breather is sent toward a well, one would expect that if the incoming speed is large enough or if the well is small and shallow, the breather should be able to go through it and emerge on the other side of the well. We have found that this is indeed the case, but the picture is more complicated than for the scattering of a kink on the well. We have found that the outcome of the scattering process is very sensitive to the phase of the breather. While at large speeds the breather seems to always pass through the well, it is also true that for most values of the well's width and depth, the soliton can also pass through the well for nearly any value of the speed if it has the right phase.

It is important to stress that the scattering is always inelastic. When the breather crosses the well it always radiates away some energy. The energy of the outgoing breather is thus always smaller than the initial energy. The radiated energy can come from two different sources: the translational kinetic energy of the breather or its internal energy. We have observed that indeed both energies can decrease but more surprisingly we have also seen that one can increase while the other one decreases, with the total, of course, decreasing. In particular, we have observed that sometimes the translational kinetic energy of the breather increases during the scattering. In these cases, the well thus acts like a slingshot. This is illustrated in Fig. 1(a) where we present the position of the maximum of the energy density as a function of time. The oscillations are caused by the vibration of the breather and we clearly see that the speed of the breather jumps from $v=0.1$, before the scattering, to $v=0.17$ afterwards. The period of oscillation also changes from $T=62.517$ to $T=31.87$, corresponding to $\omega=0.1$ and $\omega=0.194$, respectively.

In Fig. 1(b) we present the outgoing speed of the breather as a function the breather's initial position (i.e., its phase) for the case $L=10$, $a=0.2$, and $v=\omega=0.1$. Notice that in this case the outgoing speed of the breather is nearly always larger than its initial speed. This is a generic feature we have observed for $v=\omega=0.1$. When we took $v=\omega=0.3$ we noticed that the outgoing speed tended to oscillate typically in the range [0.2, 0.4]. Looking at Fig. 1(b), we also observe a small amount of backward scattering described below.

More surprisingly we have seen a few cases of a breather being split in two breathers [Fig. 1(c)]. One is ejected from the well while another one remains trapped inside it. This leads us to another observed scattering outcome: the trapping of the breather.

B. Breather trapping

When a breather scatters on a well, it can become trapped in it. As we will see later, this occurs more often when the well is deep. Once trapped the breather is actually unstable and it slowly radiates away its energy and the breather's internal parameter ω slowly grows toward $\omega=1$. This happens because of the perturbation introduced by the well.

In the small amplitude limit, Eq. (4) becomes

$$f_{tt}-f_{xx}+k(1-\alpha)f=0, \tag{6}$$

which can be solved by taking $f(x,t)=\sin(\omega t)g(x)$. The resulting Sturm-Liouville problem can be solved as described in [15] to obtain

$$g(x) = \begin{cases} A\cos\left(\frac{L\lambda}{2}\right)\exp[\sqrt{ka-\lambda^2}(x+L/2)] & x<-L/2 \\ A\cos(\lambda x) & -L/2\le x\le L/2 \\ A\cos\left(\frac{L\lambda}{2}\right)\exp[\sqrt{ka-\lambda^2}(L/2-x)] & L/2<x, \end{cases} \tag{7}$$

where λ is determined by solving the transcendental equation

$$\cos\left(\frac{\lambda L}{2}\right)=\frac{\lambda}{\sqrt{ka}} \tag{8}$$

and $\omega^2=\lambda^2+k(1-a)$. The period of oscillation is then given by $\tau=\frac{2\pi}{\sqrt{\lambda^2+k(1-a)}}$.

In the limit $2/(L\sqrt{ka})\gg 1$, we have $\lambda\approx\sqrt{ka}$ and $\tau\approx 2\pi/k$ which matches the frequency of a small amplitude breather. The ground state obtained in the small-amplitude limit and the small-amplitude breather are thus very similar and can be thought of as sine-Gordon vibrations in the well derived in two different limits. The breathers are vacuum excitations of the sine-Gordon model. When confined in a large well they can have any width for as long as they fit inside the well. The amplitude of the breather is linked to its width and frequency. The linear vibrations in the well, on the other hand, have a fixed size and period, but their amplitude is arbitrary for as long as they remain small. In a relatively narrow well, the breather does not really fit inside the well and so becomes a linear vibration of large amplitude.

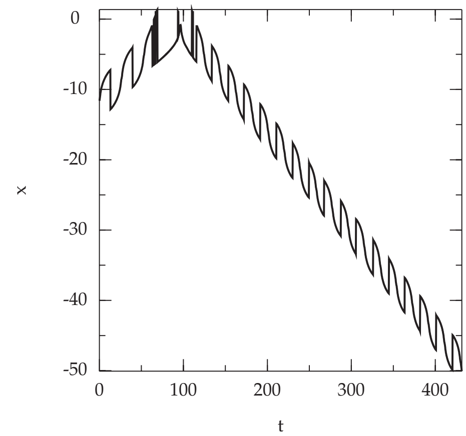

FIG. 2. Position of the breather in the case of a well centered on $x=0$ for $L=2$, $a=0.2$, $v=\omega=0.12$, and $x_0=-11.6$.

The sine-Gordon vibrations in the well always decay by radiating away some energy: the *linear vibrations* radiate because of the nonlinearity of the sine-Gordon equation while the breather radiates because of the perturbation introduced by the well.

As stated in the previous section we have sometimes observed a breather going through the well but leaving a relatively large oscillation in it. In Fig. 1(c) the amplitudes of oscillations for the excitations inside the well and the outgoing breather are 1.7 and 2.7, respectively, [this is not visible in Fig. 1(c) simply because the two oscillations are out of phase and we have chosen to plot the figure at the time when both excitations are relatively large]. The formation of a double breather is very rare though; out of 100 000 or so simulations that we have performed, we have only seen the creation of a double breather a few times in the regions of parameters corresponding to the transition between the trapping and the transmission of breathers.

C. Breather backward scattering

The breather can sometimes bounce out of the well. As we will show later, this tends to occur mostly at small speeds and in a narrow and shallow well, but unlike what happens for the scattering for the kink, this scattering mode is observed for large ranges of the parameters values. As for the forward scattering, the breather can be ejected backward from the well with a speed larger than it had initially. This is seen in Fig. 1(b) in two narrow regions of x_0 where the outgoing speed of the breather is negative.

In Fig. 2 we present the time evolution of the position of the breather during a backward scattering for the case $L=2$, $a=0.2$, $v=\omega=0.12$, and $x_0=-11.6$. In this case the outgoing velocity is $v_{\rm out}=-0.144$.

The breather backward scattering happens to be the dominant scattering mode when $\omega=2\pi/\tau$ is small. This corresponds to the limit where the breathers have a small amplitude and, taking $k=1$, a frequency τ slightly larger than 2π. We can study the scattering of such breathers by computing the transmission and reflection coefficients of plane waves

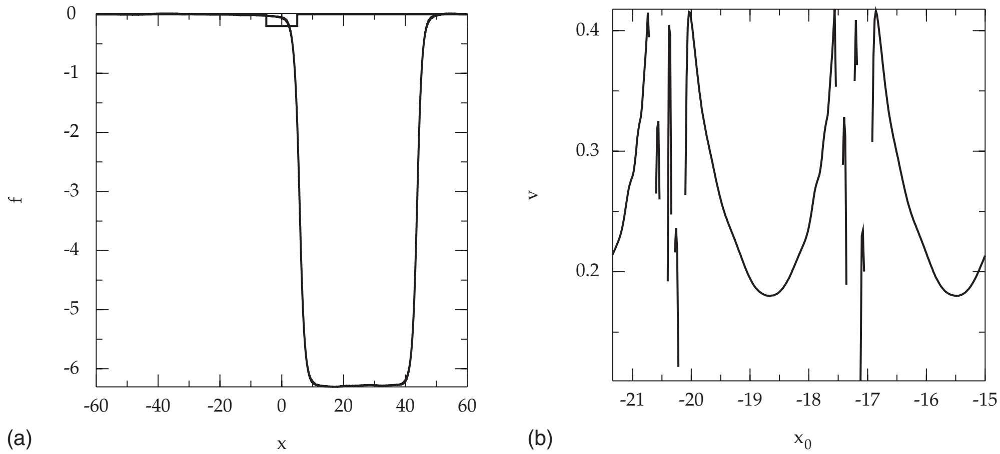

FIG. 3. Split breather scattering with $L=10$, $a=0.2$, $v=\omega=0.14$. (a) Profile after the scattering for $x_0=-17.12$. (b) Outgoing speed of the kink or antikink (the gaps correspond to the regions of forward scattering) as a function of the initial position (phase) x_0.

for the linear Eq. (6). Such solutions can be written in the complex form as

$$g(x)=\begin{cases} e^{i(\omega t-\lambda x)}+Re^{i(\omega t+\lambda x)} & x<-L/2 \\ \mathcal{T}e^{i(\omega t+\lambda x)} & L/2<x, \end{cases} \tag{9}$$

where

$$\lambda^2=\omega^2-1,$$

$$\lambda_0^2=\omega^2-(1-a),$$

$$R=\frac{4\lambda\lambda_0 e^{iL(\lambda-\lambda_0)}}{(\lambda+\lambda_0)^2+(\lambda-\lambda_0)^2e^{-2iL\lambda_0}},$$

$$\mathcal{T}=\frac{(\lambda^2-\lambda_0^2)e^{iL\lambda}(1+e^{-2iL\lambda_0})}{(\lambda+\lambda_0)^2+(\lambda-\lambda_0)^2e^{-2iL\lambda_0}}. \tag{10}$$

In the limit $\lambda\to 1$ and if $\lambda \ll \lambda_0$, we have $R\approx -1$ and $\mathcal{T}\approx 0$. This shows that a linear superposition of small amplitude waves with a frequency $\tau\approx 2\pi$ gets reflected by the well and explains why small amplitude breathers get reflected by the well too.

D. Breather splitting

The most interesting phenomenon we have observed when scattering a breather on the well is the spitting of the breather into a kink and antikink pair. As the energy of a breather or a kink decreases when they fall into the well, the excess of their energy can be used to split the breather into a kink–antikink pair. One of them remains inside the well while the other escapes from it and moves backward or forward. Initially, we expected this phenomenon to occur only in a very narrow region of the parameter space. It has turned out, however, that together with the transmission of a breather through the well, this is the most frequent outcome of the scattering process. The kink or antikink can be ejected from the well in either direction but, as will be shown later, the forward scattering is more frequent, especially for shallow and wide wells.

As the energy of a breather of frequency ω and speed v outside the well is $E_{\rm br}=16\sqrt{k}\sqrt{(1-\omega^2)}/\sqrt{1-v^2}$ and the energy of a kink trapped inside the well assumed to be large enough to contain the kink, together with an antikink outside it is $E_{\rm kak}=8\sqrt{k}(1+\sqrt{1-a})$ we can easily evaluate the critical speed below which the splitting would be impossible,

$$v_c=\left[1-\frac{4(1-\omega^2k)}{(1+\sqrt{1-a})^2}\right]^{1/2}. \tag{11}$$

When $v\le\omega\sqrt{k}$, we find that $v>v_c$ and so we see that the splitting is always possible from an energetic point of view.

Notice that the trapping of a kink and the ejection of an antikink is equivalent to the trapping of an antikink and the ejection of a kink. To see this, we observe that if we multiply $f(x,t)$ by -1, a kink becomes an antikink and vice versa, while the breather becomes a breather with the opposite phase. So any solution with a trapped kink and an ejected antikink can be transformed into a solution with a trapped antikink and an ejected kink by changing the phase of the breather by 180 degrees.

The scattering of the breather on the well is inelastic and generates some radiation waves so not all the trapping energy is transferred to the ejected kink (antikink). We have also observed that after the scattering, the slopes of the kink and antikink oscillate a little. As the sine-Gordon kink does not have genuine vibration modes [16] this extra energy is then slowly radiated away. Moreover, the trapped kink (antikink) moves back and forth inside the well even when the well is quite narrow.

In Fig. 3 we show the profile of a split breather after its scattering on a well with $L=10$, $a=0.2$, $v=\omega=0.14$, and $x_0=-17.12$. The speed of the ejected kink in this case is $v_{\rm kink}=0.109537$. The speed of the ejected kink or antikink varies with the phase of the breather. This is shown in Fig. 3(b) where we present the outgoing speed of the kink after the scattering. Note that the speed of the outgoing kink (antikink) is nearly always larger than the incoming speed of the

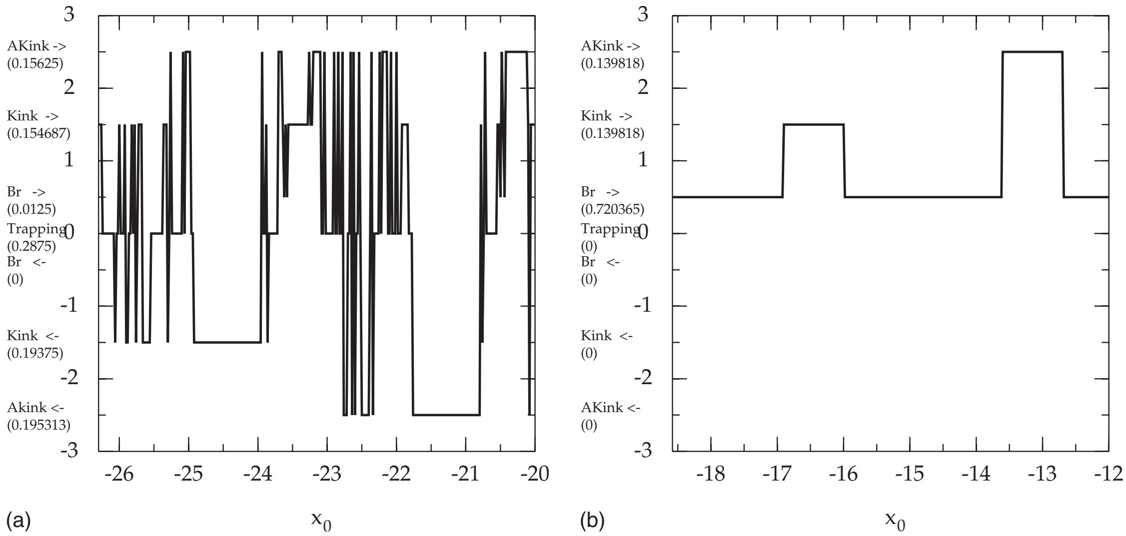

FIG. 4. Phase dependence of the scattering modes for a breather as a function of the initial position (phase) x_0 (a) $a=0.5$, $L=20$, $v=\omega=0.1$; (b) $a=0.2$, $L=2.4$, $v=\omega=0.3$. “Br->” and “Br<-:” forward and backward breather scattering. “Kink->” and “Kink<-:” ejection of a forward or backward kink. “AKink->” and “AKink<-:” ejection of a forward or backward antikink. The number below the label corresponds to the relative occurrence of this scattering mode as the phase varies.

breather. This is true in most cases but it depends a little on the depth and the width of the well.

III. PARAMETER DEPENDENCE

The most important parameter in determining the properties of the scattering of the breather on the square well is the phase of the breather. In many instances, the scattering outcome is very sensitive to its value. This is shown in Fig. 4 where we present the scattering mode as a function of the breather phase (i.e., the breather's initial position) in two extreme cases.

In Fig. 4(a), we have $a=0.5$, $L=20$, and $v=\omega=0.1$, and we see that the scattering mode is very sensitive to the breather phase. This is very common for small values of v and ω, especially when most scattering modes can occur. The numbers in parentheses below the mode names correspond to the total fraction of these modes for the full range of the phase.

In Fig. 4(b), we have $a=0.2$, $L=2.4$, and $v=\omega=0.3$, and now the scattering is very regular: the breather splits into a kink and antikink, with one of them trapped and the other one sent forward, in two well-defined phase regions and it scatters forward in other cases. We have observed this type of pattern especially when the speed is large. Our explanation for this is that when the breather moves slowly, it has time to oscillate several times inside the well. Its vibration phase thus changes relatively rapidly during the scattering process, leading to different scattering modes. At large speeds, on the other hand, the breather mostly goes through the well and scatters with the phase it has at that time. This fits well with the observations we have made when studying the scattering of a baby-Skyrme soliton on a square well [17] where at low speeds, the soliton oscillates several times inside the well before emerging from it on one side of the well or the other.

A. Dependence on v and ω

By letting $v=\omega$, for the breather, its energy is exactly equal to the energy of a kink and an antikink infinitely separated. Thus this is the critical value for the breather to be able to split into a pair of a kink and an antikink outside the well and as such it is a natural choice of parameters to study the scattering modes of the breather. We will consider what happens when $v\neq\omega$ in a later section.

In Fig. 5, we present the relative occurrence of the different scattering modes as a function of the speed $v=\omega$ for two different wells: $a=0.2$, $L=2$ [Fig. 5(a)] and $a=0.2$, $L=10$ [Fig. 5(b)]. For small values of v and ω, the breather dominantly splits into a kink and an antikink with one of them trapped inside the well while the other escapes backward. This is a general feature at low speeds when the well is narrow and reasonably deep. Looking at movies that we have made of several scatterings of this type, we have always observed the following: when the breather hits the well, the kink (antikink) falls into the well and gets trapped. Because of the narrowness of the well, the remaining antikink (kink) is neither able to fall inside it nor to push the kink (antikink) outside the well. It thus has no other choice but to bounce on the trapped kink (antikink). If the initial speed is increased sufficiently, the second antikink (kink) has enough energy to push the trapped kink (antikink) at least partially out of the well. This can then result in a forward or backward scattering as well as in the splitting of the kink where a kink or an antikink is ejected forward.

When the speed $v>0.15$, there are only two scattering modes: forward transmission or forward splitting. Note the oscillations between these two modes in Fig. 5(b). When $v>0.4$, there is always only one scattering mode left, the transmission of the breather. This can be easily explained by the fact that the breather has enough energy to cross the well quickly without being affected much by it.

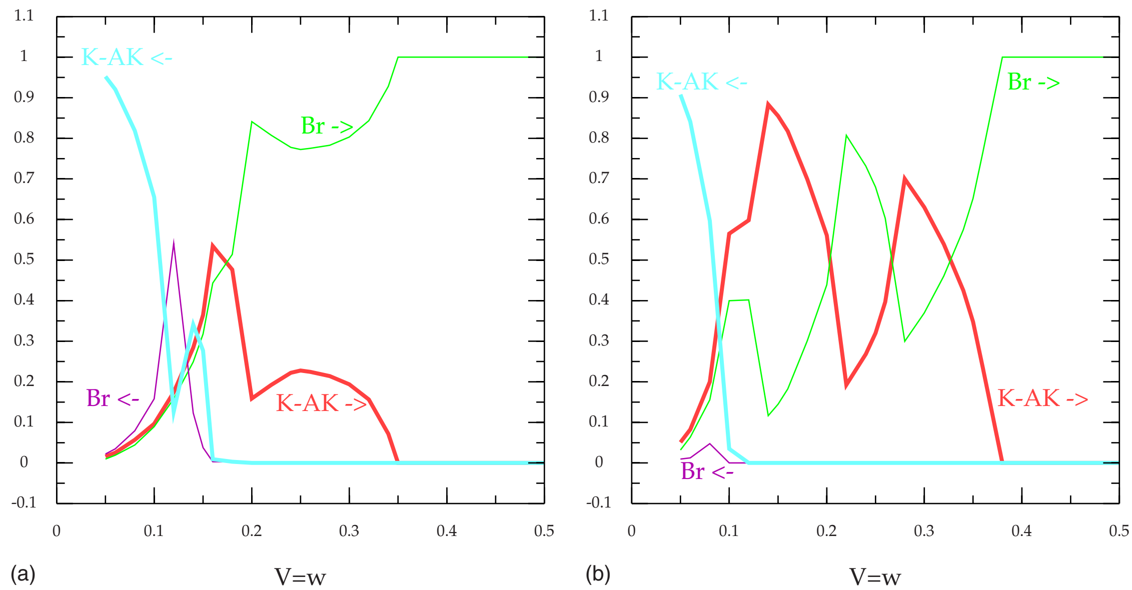

FIG. 5. (Color online) Scattering mode frequencies as a function of $v=\omega$. (a) $a=0.2$, $L=2$; (b) $a=0.2$, $L=10$. "Br-<" (thin purple/dark gray): forward breather scattering. "Br<-" (thin green/light gray): backward breather scattering. "K-AK->" (thick red/dark gray): trapped kink (antikink) and forward antikink (kink). K-AK->" (thick blue/light gray): trapped kink (antikink) and backward antikink (kink).

B. Dependence on the well width *L*

The width of the well also affects the scattering of the breather, as is shown in Fig. 6(a) for $a=0.2$, $v=\omega=0.1$ and in Fig. 6(b) for the cases $a=0.2$, $v=\omega=0.3$.

When $v=\omega=0.1$ and for very narrow wells, the dominant scattering mode is the forward splitting of the breather. The well is so narrow that the second kink has enough energy to push the first one out of the well. When the well is wider than 2 the dominant mode is the backward splitting of the breather as explained in Sec. II C.

Once the well is larger than 10, only 2 scattering modes are relevant: the forward transmission and the forward splitting. What we have observed when the well is very large is that when the breather enters the well it splits into a kink and an antikink pair. For a reason we have not understood yet, whichever of the kink or antikink is in front happens to have more energy than its partner. The kink and the antikink then move forward, slowly increasing the distance separating them until one of them hits the other side of the well. As it has more energy, it manages to climb out of the well and to

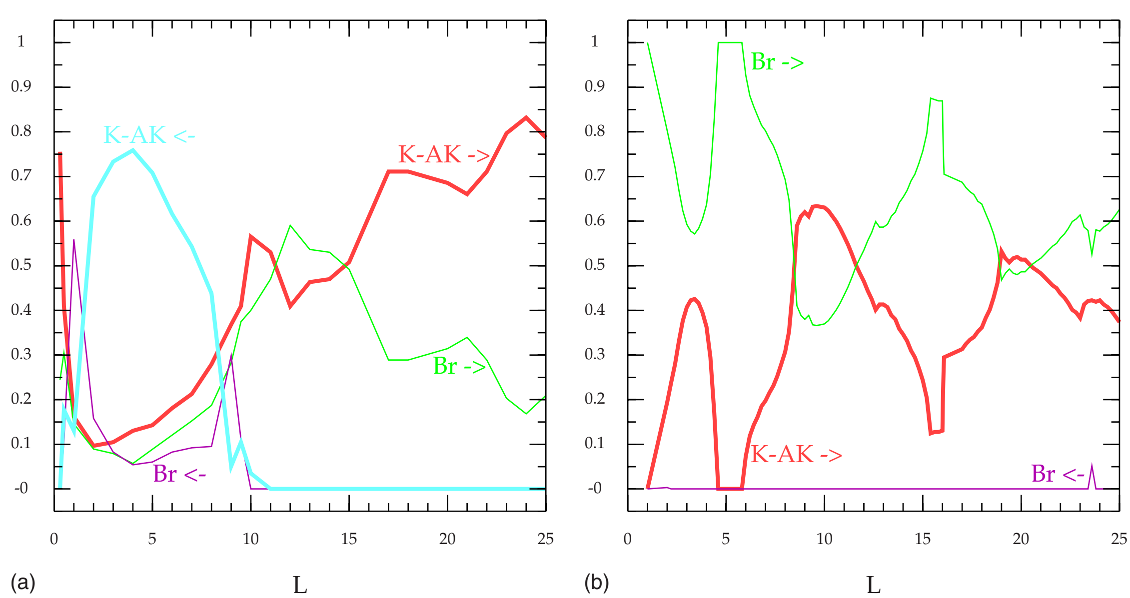

FIG. 6. (Color online) Scattering mode frequencies as a function of L. (a) $a=0.2$, $v=\omega=0.1$; (b) $a=0.2$, $v=\omega=0.3$ "Br-<" (thin purple/dark gray): forward breather scattering. "Br<-" (thin green/light gray): backward breather scattering. "K-AK->" (thick red/dark gray): trapped kink (antikink) and forward antikink (kink). "K-AK->" (thick blue/light gray): trapped kink (antikink) and backward antikink (kink).

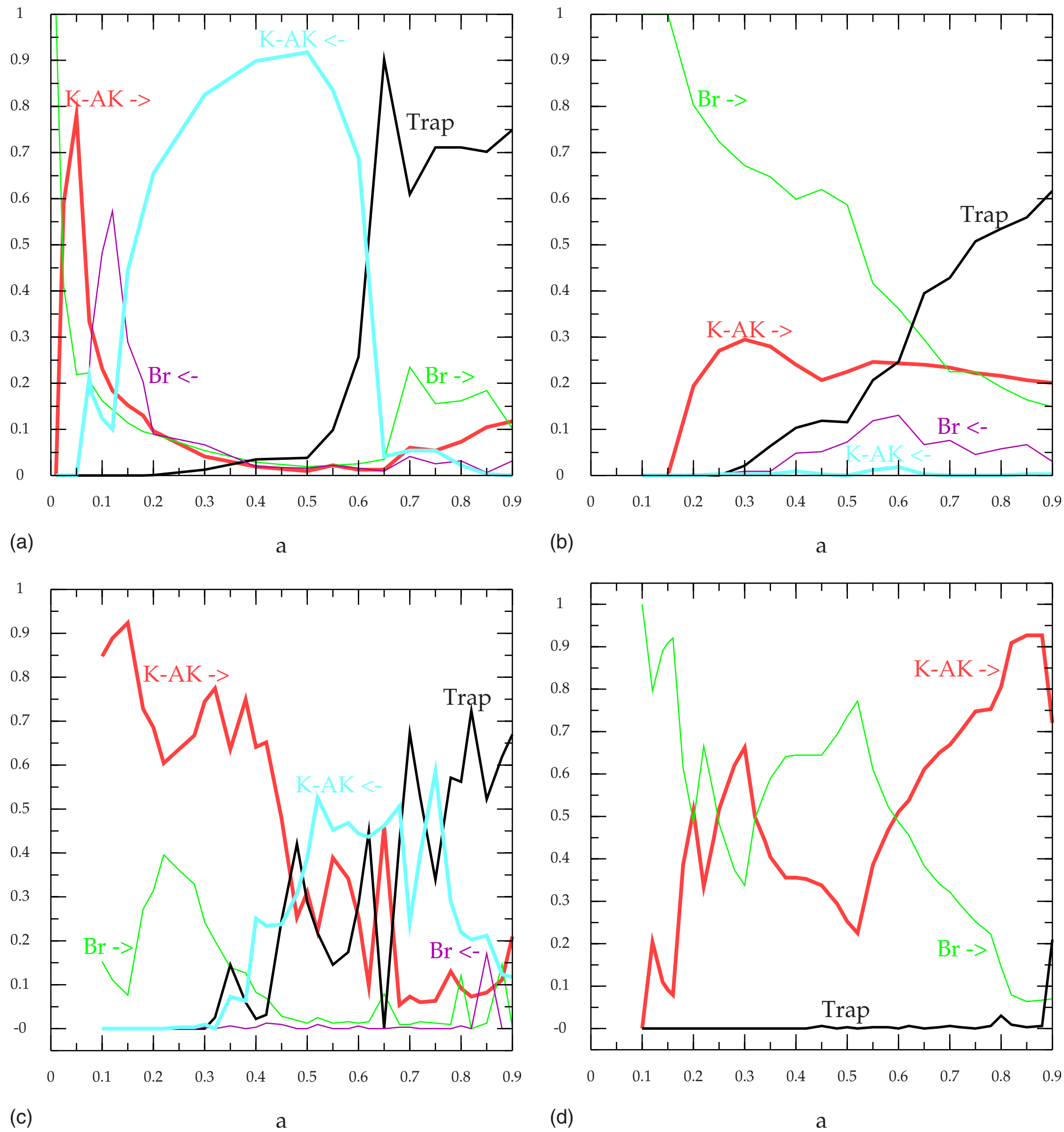

FIG. 7. (Color online) Scattering mode frequencies as a function of a. (a) $L=2$, $v=\omega=0.1$; (b) $L=2$, $v=\omega=0.3$; (c) $L=20$, $v=\omega=0.1$; (d) $L=20$, $v=\omega=0.3$; “Br-<” (thin purple/dark gray): forward breather scattering. “Br<-” (thin green/light gray): backward breather scattering. “Trap” (black): trapped breather. “K-AK->” (thick red/dark gray): trapped kink (antikink) and forward antikink (kink). “K-AK->” (thick blue/light gray): trapped kink (antikink) and backward antikink (kink).

escape forward while its partner, with less energy, remains trapped inside the well.

The phase of the breather when it falls into the well determines which of the kink or the antikink, into which the breather has divided, is at the front and eventually escapes from the well. In the intermediate region between the two modes, there is a succession of narrow windows where the breather goes through the well, separated by windows where the kink or the antikink, into which the breather has split, escapes from the well.

C. Dependence on the well depth a

The depth of the well plays a major role in the breather scattering as this is the parameter that determines the binding energy and the size of a breather or a kink inside the well.

As shown in Fig. 7(a), for a narrow well ($L=2$) and at a small speed ($v=\omega=0.1$), the scattering modes vary greatly with the depth of the well. For very shallow wells ($a<0.02$), the dominant mode is the forward transmission: the breather hardly sees the well at all. For marginally deeper wells ($0.02<a<0.15$), the dominant modes are the forward splitting and then the backward scattering. For deeper wells ($0.15<a<0.6$), as explained in Sec. II C, the backward splitting dominates. Then when ($a>0.6$), the well is so deep that the breather is usually trapped in it.

When the speed is increased to $v=\omega=0.3$, see Fig. 7(b) which looks roughly like a stretched out version of Fig. 7(a): for a very small a the only scattering mode is, as one might expect, the forward scattering; as a increases, the ejection of

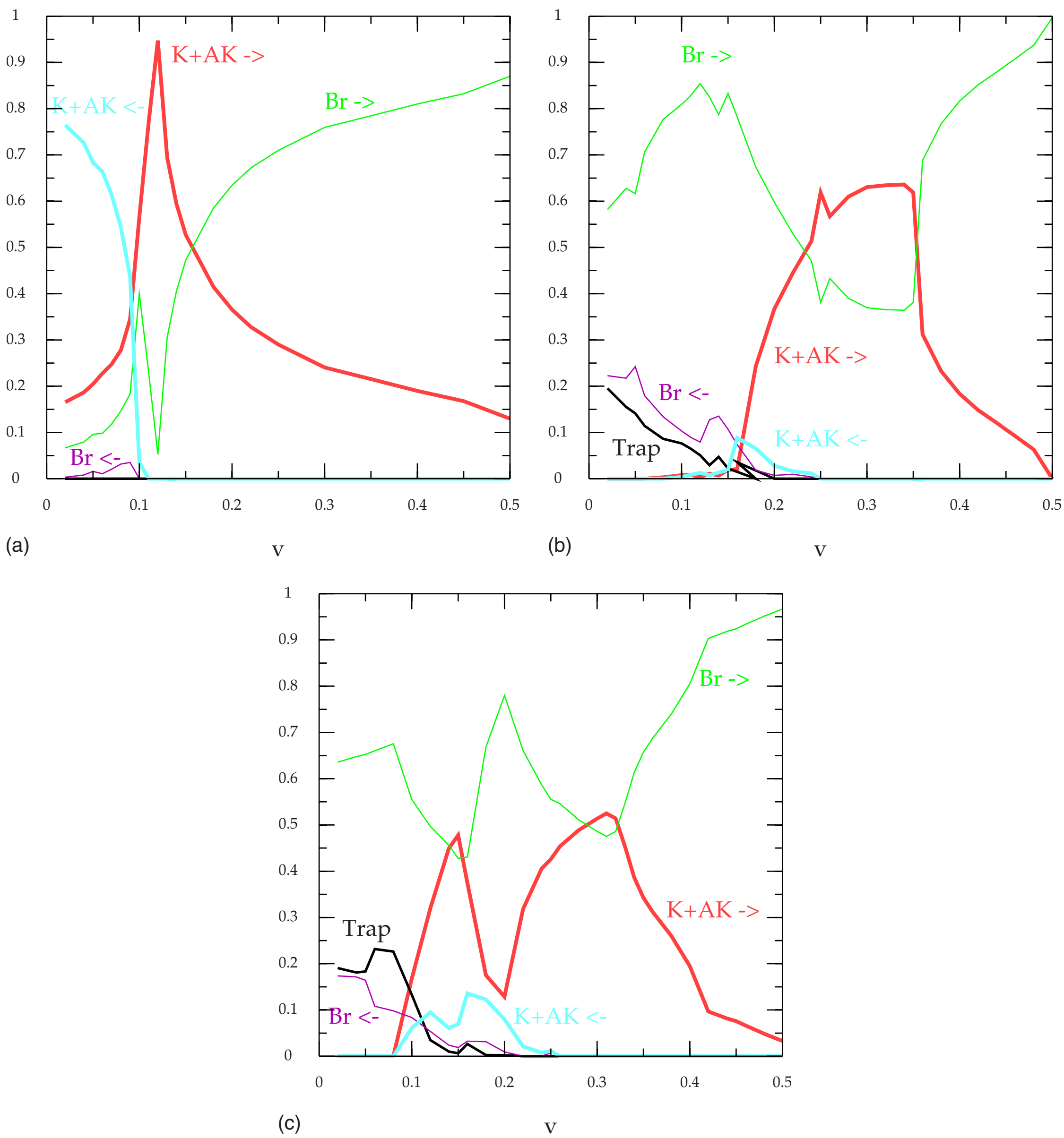

FIG. 8. (Color online) Scattering mode frequencies as a function of v for $a=0.2$ and (a) $L=10$, $\omega=0.1$; (b) $L=10$, $\omega=0.3$; (c) $L=20$, $\omega=0.3$; "Br-<" (thin purple/dark gray): forward breather scattering. "Br<-" (thin green/light gray): backward breather scattering. "Trap" (black): trapped breather. "K-AK->" (thick red/dark gray): trapped kink (antikink) and forward antikink (kink). "K-AK->" (thick blue/light gray): trapped kink (antikink) and backward antikink (kink).

a kink or an antikink starts to occur; as a increases further the trapping of the breather starts to happen too. The main difference between Figs. 7(a) and 7(b), apart from the very approximate stretching, is that the backward kink–antikink mode hardly occurs at all when $v=\omega=0.3$ [Fig. 7(b)]. When a is very large, the trapping mode dominates. It is also interesting to note in Fig. 7(a) the sharp sudden transition between the backward kink–antikink mode and the trapping mode just over the value $a=0.6$. We do not have any explanation for it.

For wide wells, the picture changes significantly. As explained in Sec. II D, the breather nearly always splits into a kink–antikink pair inside the well. For the case presented in Fig. 7(c), $L=20$ and $v=\omega=0.1$, the parameters are such that the forward splitting is dominant for ($a<0.45$). For deeper wells, the scattering mode changes very rapidly as a increases, but overall, the breather trapping dominates. In the region between $a=0.5$ and $a=0.8$ we observe very sharp oscillations between the trapping and the backward-scattering modes. We believe that these transitions are generated by very subtle phase effects between the kink and the antikink in the well.

When the speed is increased to $v=\omega=0.3$, see Fig. 7(d), the two dominant modes are the forward splitting and the forward transmission. Trapping occurs rarely, only for very deep wells. In this case, the breather has enough energy to interact with the well quickly and, at least in part, escape from it. The difference between Figs. 7(c) and 7(d) is striking: while in the former figure all the modes are observed, in the latter, only two modes are seen, except for values of a

very close to 1 where the trapping starts to occur too. The speed in Fig. 7(d) is three times larger than in Fig. 7(c), explaining why the backward modes are not observed at all in Fig. 7(d).

IV. VARYING v AND ω SEPARATELY

So far we have looked only at the scattering of a breather on a well for the special case $v=\omega$, that is when the breather has exactly the same energy as an infinitely separated pair of a kink and an antikink. This critical case is particularly interesting, but it is also very interesting to investigate what happens when v and ω differ for each other.

The results are shown in Fig. 8 where we have taken $a=0.2$ for the depth of the well and we have considered a well of widths $L=10$ and $L=20$ for $\omega=0.1$ and $\omega=0.3$.

Looking at the figures we note that when $v \gg \omega$, the breather is dominantly going through the well but the forward ejection of a kink or an antikink also occurs, but to a lesser extent. This confirms what we have seen so far, that when the speed is large, i.e., larger than 0.5, the soliton has so much energy that it does not feel the well too much and it just passes through it.

When $v<0.1$, the dominant mode is always the forward transmission of the breather, though the other modes occur too. This is somewhat harder to explain.

In the three cases we have considered, the largest rates of kink–antikink splitting occur in the region where $v \approx \omega$, the actual maximum being reached when v is slightly larger than ω. In the last figure, which corresponds to a large well, we observe two maxima for the kink–antikink splitting: one around $v \approx \omega$ and one around $v \approx \omega/2$. We do not have any explanation of this behavior except that the relative phase of the bound kink and antikink plays a very important role in the scattering with the well.

When v is smaller than ω, all the familiar modes, i.e., trapping, backward scattering of the breather, and backward kink–antikink splitting, can all occur but at a relatively small rate. This can be explained by the fact that having less kinetic energy, the breather is more likely to bounce on the far side of the well and, if the phase is right, come out of the well.

The fact that the trapping of the breather does not occur for a very small speed more often is an indication that the breather is fairly robust and that at a small speed, the scattering is nearly adiabatic, leading to the forward scattering.

V. CONCLUSIONS

In this paper, we have shown that the scattering of a breather on a square well exhibits very interesting phenomena. The breaking of the sine-Gordon integrability due to this inhomogeneity leads to scattering modes that are forbidden in an integrable model. In particular the sine-Gordon breather can be split into a kink and an antikink which move both forward and backward. Somewhat surprisingly, this scattering mode is genuine and sometimes, depending on the parameters of the model, the dominant one.

Another surprising phenomenon seen in the scattering is that the well can accelerate the breather. This is possible because the internal energy of the breather can be partly converted into its translational kinetic energy. This acceleration can occur for the breather transmission as well as the breather reflection.

We solved our Eq. (4) in dimensionless units to make our results as widely applicable as possible. The main changes for the description of a physical system would be the actual size of the well, breathers, and the kink. The width of the well would always be expressed in the same units as the width of the kink (≈ 5 in our units), while the depth of the well would be set by a which is a dimensionless parameter. The breather parameter ω is also a dimensionless parameter. The speed of the kinks and breathers were all evaluated in units of the wave speed of the sine-Gordon equation (1 in our units). Thus to interpret our results in a particular physical application, one would just need to know the size of the kink in the system and the speed of the small amplitude waves.

The parameter dependence of the scattering data is quite complicated. It is also very sensitive to the phase of the breather when it collides with the well. Overall, we have observed that at high energies, the breather tends to scatter forward or to split forward into a kink–antikink pair while at low energies, on the other hand, all the scattering modes can take place.

Given our observations, it would be interesting to find out if one can reverse the scattering process and create a breather by the scattering of a kink on a trapped antikink (or viceversa). If this process is possible, then it would provide a method to experimentally generate a breather from a kink and an antikink. We plan to investigate this in the future.

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